TWMS J. Pure Appl. Math. V.9, N.1, 2018, pp.3-17

GENERALIZED METRIC SPACES: SURVEY

TATJANA DOŠENOVIĆ[1], STOJAN RADENOVIĆ[2], SHABAN SEDGHI[3]

Abstract. In this paper a review of some well known generalizations of metric spaces is given. Special attention is focused on on a detailed overview of a relatively new generalization of metric spaces, the so-called S-metric spaces, new fixed point results as well as theirs applications.

Keywords: S-metric space, fixed point, continuous, nondecreasing, lower semi-continuous mapping.

AMS Subject Classification: Primary 47H10; Secondary 54H25.

1. Introduction

We start our paper by recalling the basic metric spaces.

Definition 1.1. *Let $X \neq \emptyset$ and $d : X \times X \to [0, \infty)$ be such that:*
$M_1)$ $d(x, y) = 0$ *if and only if* $x = y$,
$M_2)$ $d(x, y) = d(y, x)$ *for all* $x, y \in X$,
$M_3)$ $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.

Remark 1.1. *If in Definition 1.1 instead of $M_3)$ we have $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ for all $x, y, z \in X$ the space is called ultra-metric space.*

Very interesting paper in the sense of ultra-metric space is [14].

Theorem 1.1. *Every metric space is Hausdorff.*

Let $(X_1, d_1), (X_2, d_2), \ldots, (X_n, d_n)$ be a metric spaces. Then we can define a product of metric spaces in the following way:

$$X_1 \times \cdots \times X_n = \{(x_1, \ldots, x_n),\ x_i \in X_i,\ i = 1, 2, \ldots, n\}.$$

Now we can define a mapping D as follows:

$$D((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max_{1 \leq i \leq n} d_i(x_i, y_i),$$

for every $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in X_1 \times \cdots \times X_n$.

Theorem 1.2. *D is a metric on $X_1 \times X_2 \times \cdots \times X_n$.*

Proof. We shall prove that all of the conditions $M_1 - M_3$ are satisfied.

$M_1)$ $D((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = 0$ is equivalent to $d_i(x_i, y_i) = 0$ for every $i = 1, \ldots, n$. Since d_i is a metric we have that $x_i = y_i$ for every $i = 1, \ldots, n$, and so $(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$.

[1] Faculty of Technology, University of Novi Sad, Serbia, e-mail: tatjanad@tf.uns.ac.rs
[2] University of Belgrade, Faculty of Mechanical Engineering, Serbia, e-mail: radens@beotel.rs
[3] Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran
e-mail: sedghi_gh@yahoo.com, sedghi.gh@qaemiau.ac.ir

Manuscript received September 2017.

M_2) Symmetry follows from

$$D((x_1,\dots,x_n),(y_1,\dots,y_n)) = \max_{1\le i\le n} d_i(x_i,y_i) = \max_{1\le i\le n} d_i(y_i,x_i) = D((y_1,\dots,y_n),(x_1,\dots,x_n)).$$

M_3)

$$D((x_1,\dots,x_n),(y_1,\dots,y_n)) = \max_{1\le i\le n} d_i(x_i,y_i) \le \max_{1\le i\le n}(d_i(x_i,z_i)+d(z_i,y_i))$$

$$\le \max_{1\le i\le n} d_i(x_i,z_i) + \max_{1\le i\le n} d(z_i,y_i)$$
$$= D((x_1,\dots,x_n),(z_1,\dots,z_n)) + D((z_1,\dots,z_n),(y_1,\dots,y_n)).$$

Therefore, the product of metric spaces (X_1,d_1), $(X_2,d_2),\dots,(X_n,d_n)$ is a metric space $(X_1\times X_2\times\cdots\times X_n, D)$. □

We can define a product of metric spaces on one more way: If $X=(x_1,\dots,x_n)$ and $Y=(y_1,\dots,y_n)$ and $D_S(X,Y)=\sum_{i=1}^n |x_i-y_i|$ then $(X\times Y, D_S)$ is a metric space too.

A first generalization of a metric spaces is given in the following definition.

Definition 1.2. *Let X be a set. A mapping $d: X\times X\to[0,\infty)$ is called a pseudodistance if and only if for any $x,y,z\in X$, we have*

m_1) $d(x,x)=0$,

m_2) $d(x,y)=d(y,x)$,

m_3) $d(x,y)\le d(x,z)+d(z,y)$.

The pair (X,d) is called a pseudometric space.

Obviously every metric space is a pseudometric space. A pseudometric space is a generalized metric space in which the distance between two distinct points can be zero. Pseudometric spaces are not necessarily Hausdorff. The difference between pseudometrics and metrics is entirely topological. That is, a pseudometric is a metric if and only if the topology it generates is T_0; (i.e. distinct points are topologically distinguishable). By using equivalence relations xy if d(x, y) = 0 pseudometric space is a metric space.

Let (X_i,d_i), $i=1,2,\dots,n$ be a pseudometric, and let $X=(x_1,\dots,x_n)$, $Y=(y_1,\dots,y_n)$ be points in the product $X_1\times X_2\times\cdots\times X_n$. Then

$$D_s(X,Y)=\sum_{i=1}^n d_i(x_i,y_i) \text{ and } D_m(X,Y)=\max\{d_i(x_i,y_i): i=1,2,\dots,n\}.$$

The next generalization is a quasi metric space.

Definition 1.3. *Let X be a set. A mapping $d: X\times X\to[0,\infty)$ is called a quasi-distance if and only if for any $x,y,z\in X$, we have*

Q_1) $d(x,y)=0$ *if and only if* $x=y$,

Q_2) $d(x,y)\le d(x,z)+d(z,y)$.

In this case, the pair (X,d) is called a quasi-metric space.

In [19] the authors claim that every quasi-metric space is also a metric space. But in [10] an example of a quasi-metric is given which is not a metric.

Example 1.1. *Let $X=\mathbb{R}$ and d be defined by* $d(x,y)=\begin{cases} x-y, & x\ge y\\ 1, & x<y\end{cases}$.

Then d is a quasi-metric on X but d is not a metric on X.

Let (X,d) be a quasi-metric space. Since d satisfies all of the properties of a metric space except the triangle inequality, we can define a map $d*: X\times X\to[0,\infty)$, such that $d*(x,x*)=\inf\sum d(x_i,x_{i+1})$, where the infimum is taken over all sequences $x=x_0,...,x_{n+1}=x*$ in X. Therefore, $d*$ satisfies the triangle inequality. But, the problem with this approach is that $d(x,x*)$ could be 0 for different points $x,x*$ and condition (M1) is not satisfied for $(X,d*)$.

If (X,d) is a quasi-metric space, then we do not necessarily have uniqueness of the limit of a sequence.

Definition 1.4. [49] *Let X be a set. A mapping $d: X\times X\to[0,\infty)$ is called a semi-distance if and only if for any $x,y\in X$, we have*

Sd_1) $d(x,y)=0$ if and only if $x=y$;

Sd_2) $d(x,y)=d(y,x)$.

In this case, the pair (X,d) is called a semi-metric space.

Definition 1.5. [3, 6] *Let X be a nonempty set and $s\geq 1$ be a given real number. A function $d: X\times X\to[0,\infty)$ is a b-metric on X if, for all $x,y,z\in X$, the following conditions hold:*

b_1) $d(x,y)=0$ if and only if $x=y$,

b_2) $d(x,y)=d(y,x)$,

b_3) $d(x,z)\leq s[d(x,y)+d(y,z)]$ (b-triangular inequality).

In this case, the pair (X,d) is called a $b-$metric space (or metric type space).

If $s=1$, then the triangle inequality in a metric space is satisfied. However it does not hold true when $s>1$. Thus the class of $b-$metric spaces is effectively larger than that of the ordinary metric spaces. The following example illustrates the above remarks.

Example 1.2. [23] *Let $X=\{-1,0,1\}$. Define $d: X\times X\to\mathbb{R}^+$ by $d(x,y)=d(y,x)$ for all $x,y\in X$, $d(x,x)=0$, $x\in X$, $d(-1,0)=3$, $d(-1,1)=d(0,1)=1$.*

Then (X,d) is a $b-$ metric space, but not a metric space since the triangle inequality is not satisfied. Indeed, we have that

$$d(-1,1)+d(1,0)=1+1=2<3=d(-1,0).$$

It is easy to verify that $d(-1,0)\leq\frac{3}{2}(d(-1,1)+d(1,0))$, and therefore (X,d) is a $b-$ metric space with $s=\frac{3}{2}$.

On the other hand, we have the following example.

Example 1.3. [33] *Let (X,d) be a metric space and $\rho(x,y)=(d(x,y))^p$, where $p>1$ is a real number. Then (X,ρ) is a b-metric with $s=2^{p-1}$.*

Definition 1.6. [4]*Let (X,d) be a $b-$metric space, $x\in X$ and $\{x_n\}$ be a sequence in X. Then*

(i) $\{x_n\}$ converges to x if and only if $\lim\limits_{n\to\infty}d(x_n,x)=0$. We denote this by $\lim\limits_{n\to\infty}x_n=x$ or $x_n\to x$ $(n\to\infty)$.

(ii) $\{x_n\}$ is a Cauchy sequence if and only if $\lim\limits_{n,m\to\infty}d(x_n,x_m)=0$.

(iii) (X,d) is complete if and only if every Cauchy sequence in X is convergent.

Remark 1.2. [4]. *In a $b-$ metric space (X,d) the following assertions hold:*

(i) A convergent sequence has a unique limit.

(ii) Each convergent sequence is Cauchy.

(iii) In general, a b-metric is not continuous.

Remark 1.3. *For claim (iii), the reader is referred to see the Example 1.2. ([18]).*

Theorem 1.3. [1] *Let (X,d) be a $b-$metric space and suppose that $\{x_n\}$ and $\{y_n\}$ converge to $x,y\in X$, respectively. Then we have*

$\frac{1}{s^2}d(x,y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x,y)$.

In particular, if $x = y$, then $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$ we have

$\frac{1}{s}d(x,z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq sd(x,z)$.

Many mathematicians have tried to extend the concept of metric space in which it is defined distance of three or more points.

Gahler ([12], [13]) first defined $2-$metric space as follows:

Definition 1.7. *Let X be a non empty set. A real valued function d on $X \times X \times X$ is said to be a $2-$metric on X if*

2_1) *given distinct elements x, y of X, there exists an element z of X such that $d(x,y,z) \neq 0$,*

2_2) *$d(x,y,z) = 0$ when at least two of x, y, z are equal,*

2_3) *$d(x,y,z) = d(x,z,y) = d(y,z,x)$ for all $x, y, z \in X$,*

2_4) *$d(x,y,z) \leq d(x,y,w) + d(x,w,z) + d(w,y,z)$ for all $x, y, z, w \in X$.*

When d is a $2-$metric on X, then the ordered pair (X,d) is called a $2-$ metric space.

We can interpret $d(x,y,z)$ as the area of the triangle spanned by x, y and z.

Remark 1.4. *If, in the definition of a $2-$ metric, condition 2_4) is deleted, then the function d is called a semi-$2-$ metric.*

Remark 1.5. [11] *A $2-$metric is not a continuous function of its variables, whereas the ordinary metric is. Also a $2-$metric space is not topologically equivalent to an ordinary metric. Therefore there is no easy way to find relationship between results obtained in $2-$metric spaces and metric spaces. In particular, fixed point theorems on $2-$metric spaces and metric spaces may be incoherent. The following things are important in terms of a $2-$metric space:*

1. Every $2-$metric is non-negative.

2. We may assume that every $2-$metric space contains at least three distinct points.

After that Dhage in 1992 ([7]) gave another definition of space together with a function of $3-$variables.

Definition 1.8. *Let X be a nonempty set, and let $\mathbb{R}$ denote the real numbers. A function $D : X^3 \to \mathbb{R}$ satisfying the following axioms:*

D_1) $D(x,y,z) \geq 0$ for all $x, y, z \in X$,

D_2) $D(x,y,z) = 0$ if and only if $x = y = z$,

D_3) $D(x,y,z) = D(x,z,y) = \ldots$ (symmetry in all three variables),

D_4) $D(x,y,z) \leq D(x,y,a) + D(x,a,z) + D(a,y,z)$ for all $x, y, z, a \in X$. (rectangle inequality),

is called a generalized metric, or a $D-$metric on X.

$D(x,y,z)$ may be interpreted as a measure of the perimeter of the triangle with vertices at x, y and z.

If $D(x,x,y) = D(x,y,y)$ for all $x, y \in X$ then D is called a symmetric $D-$metric.

Let (X,d) be a metric space. Then Dhange gave as examples of D-metrics on X;

(d1) $D(x,y,z) = \frac{1}{3}(d(x,y) + d(y,z) + d(x,z))$,

(d2) $D(x,y,z) = \max\{d(x,y), d(y,z)d(x,z)\}$.

However, in [24], to satisfy the axioms of a $D-$metric it is not necessary that d satisfy the triangle inequality, only that it be a semi-metric.

Unfortunately, most of the claims concerning the fundamental topological properties of $D-$metric spaces are incorrect (see [24]) This claim provided inspiration for the formation of more general concept called a $G-$metric space ([25]).

Definition 1.9. *Let X be a nonempty set. Suppose that $G : X \times X \times X \to [0, +\infty)$ is a function satisfying the following conditions:*

G_1) $G(x,y,z) = 0$ if and only if $x = y = z$;
G_2) $0 < G(x,x,y)$ for all $x, y \in X$ with $x \neq y$;
G_3) $G(x,x,y) \leq G(x,y,z)$ for all $x, y, z \in X$ with $y \neq z$;
G_4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$ (symmetry in all three variables);
G_5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x, y, z, a \in X$.
Then G is called a G-metric on X and (X,G) is called a G-metric space.

They claim that these properties are satisfied when $G(x,y,z)$ is the perimeter of the triangle with vertices at x, y, and z in R^2. If a is in the interior of the triangle then G_5) is the best possible.

Definition 1.10. *A $G-$metric space is symmetric if $G(x,x,y) = G(y,y,x)$.*

It was observed that, in the symmetric case, many fixed point theorems on G-metric spaces are particular cases of existing fixed point theorems in metric spaces.

If (X,d) is an ordinary metric space, then d_1 and d_2 defined above defines a $G-$metrics on X, and for this to be so it is now necessary that d satisfy the triangle inequality. Moreover, it can be done and vice versa.

Theorem 1.4. *Every $G-$ metric space defined a metric space in the following way:*

$$d_G(x,y) = G(x,x,y) + G(y,y,x), \text{ for all } x, y \in X.$$

Remark 1.6. *Let (X,G) be a $G-$metric space and let $d(x,y) = G(x,x,y)$. Then d is a quasi-metric. Symmetry does not necessarily hold, because from the properties of G metrics that $G(x,x,y) \leq 2G(y,y,x)$, we can conclude that $d(x,y)$, in the general case, is not a metric, only a quasi-metric.*

Proposition 1.1. *Let (X,G) be a G-metric space, then the function $G(x,y,z)$ is jointly continuous in all three of its variables.*

Let (X_i, G_i), $i = 1, 2, \ldots, n$ be a G-metric spaces and let $X = \Pi_{i=1}^n X_i$. Then a natural definitions for a $G-$metrics on the product space X would be

$$G_1(x,y,z) = \max_{1 \leq i \leq n} \{Gi(x_i, y_i, z_i)\} \text{ and } G_2(x,y,z) = \sum_{i=1}^{n} G_i(x_i, y_i, z_i).$$

However, unless all of the (X_i, G_i) are symmetric, G_1 and G_2 may fail to be $G-$metrics (see Example 2 in [25]).

Recently, Shaban Sedghi et. al ([39]-[42]) modified the axioms for a $D-$metric space and defined D^*-metric spaces and proved some basic properties and some fixed point and common fixed point theorems in complete D^*-metric spaces.

Definition 1.11. *Let X be a nonempty set. A generalized metric (or D^*-metric) on X is a function, $D^* : X^3 \to [0, \infty)$, that satisfies the following conditions for each $x, y, z, a \in X$:*

1. $D^*(x,y,z) \geq 0$,
2. $D^*(x,y,z) = 0$ *if and only if* $x = y = z$,
3. $D^*(x,y,z) = D^*(p(x,y,z))$, *(symmetry) where p is a permutation function,*
4. $D^*(x,y,z) \leq D^*(x,y,a) + D^*(a,z,z)$.

The pair (X, D^*) is called a generalized metric (or D^*-metric) space.

Immediate examples of such a function are
(a) $D^*(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\}$,
(b) $D^*(x,y,z) = d(x,y) + d(y,z) + d(z,x)$.
Here, d is the ordinary metric on X.

2. Main results

By modifying $D-$metric and $G-$metric spaces S. Sedghi et al. ([35]-[38],[43], [44]) introduced the concept of an S-metric space. Namely, theirs definition is following:

Definition 2.1. *Let X be a nonempty set. An $S-$metric on X is a function $S : X^3 \to [0, +\infty)$ that satisfies the following conditions for each $x, y, z, a \in X$:*

$S_1)$ $S(x,y,z) = 0$ if and only if $x = y = z$.
$S_2)$ $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.
The pair (X, S) is called $S-$metric space.
Sedghi et al. [36] gave the following remarks.
It is easy to see that every D^*-metric is $S-$metric, but, in general, the converse is not true.
Example 2.1. *Let $X = \mathbb{R}^n$ and define*
(1) $S(x,y,z) = \|x + y - 2z\| + \|y - z\|$.
(2) $S(x,y,z) = d(x,y) + d(x,z)$, *where d is the ordinary metric on X.*
(3) Let $X = \mathbb{R}^2$ and d be an ordinary metric on X. Put $S(x,y,z) = d(x,y)+d(x,y)+d(y,z)$ for all $x, y, z \in \mathbb{R}^2$; that is, S is the perimeter of the triangle given by x, y, z. Then S is an S-metric on X.
(4) Let $\mathbb{R}$ be a real line. Then $S(x,y,z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$ is an S-metric on $\mathbb{R}$. This $S-$metric is called the usual $S-$metric on $\mathbb{R}$. Furthermore, the usual S-metric space $\mathbb{R}$ is complete (see the Definition 2.2 below).
(5) Let Y be a nonempty subset of $\mathbb{R}$. Then $S(x,y,z) = |x - z| + |y - z|$ for all $x, y, z \in Y$ is an S-metric on Y. Furthermore, if Y is a closed subset of the usual metric space $\mathbb{R}$, then the S-metric space Y is complete.

Definition 2.2. *Let (X, S) be an $S-$metric space.*
***1.** A sequence $\{x_n\} \subset X$ is said to S-converge to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$; that is, for each $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $S(x_n, x_n, x) < \varepsilon$. We write $x_n \to x$ for brevity.*
***2.** A sequence $\{x_n\} \subset X$ is called an S-Cauchy sequence if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$; that is, for each $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$, such that for all $n, m \geq n_0$ we have $S(x_n, x_n, x) < \varepsilon$.*
***3.** The S-metric space (X, S) is said to be S-complete if every S-Cauchy sequence is an S-convergent sequence.*

In the sequel we formulate as well as prove several (known) important properties for the S-metric spaces:

Lemma 2.1. *Let (X, S) be an S-metric space. Then, for each $x, y \in X$ it follows that $S(x,x,y) = S(y,y,x)$.*

Proof. Let $x \neq y$. According to (S2) we have

$$S(x,x,y) \leq 2S(x,x,x) + S(y,y,x) = 0 + S(y,y,x) = S(y,y,x).$$

Further, using the same idea, it follows that

$$S(y,y,x) \leq 2S(y,y,y) + S(x,x,y) = 0 + S(x,x,y) = S(x,x,y).$$

□

Lemma 2.2. *Let (X,S) be an S- metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then*

$$\lim_{n\to\infty} S(x_n,x_n,y_n) = S(x,x,y).$$

Proof. According to (S2) we have

$$\begin{aligned} S(x_n,x_n,y_n) &\leq 2S(x_n,x_n,x) + S(y_n,y_n,x) \\ &\leq 2S(x_n,x_n,x) + 2S(y_n,y_n,y) + S(x,x,y). \end{aligned}$$

On the other hand, also using (S2), we obtain

$$\begin{aligned} S(x,x,y) &\leq 2S(x,x,x_n) + S(y,y,x_n) \\ &\leq 2S(x,x,x_n) + 2S(y,y,y_n) + S(x_n,x_n,y_n). \end{aligned}$$

Finally, we have

$$-2S(x_n,x_n,x) - 2S(y_n,y_n,y) \leq S(x_n,x_n,y_n) - S(x,x,y) \leq 2S(x_n,x_n,x) + 2S(y_n,y_n,y),$$

or, equivalently,

$$|S(x_n,x_n,y_n) - S(x,x,y)| \leq 2(S(x_n,x_n,x) + S(y_n,y_n,y)) \to 0, \text{ when } n \to \infty.$$

□

Lemma 2.3. *Let (X,S) be an S- metric space. If there exist sequences $\{x_n\}$ in X such that, for every $n \in \mathbb{N}$*

$$S(x_n,x_n,x_{n+1}) \leq lS(x_{n-1},x_{n-1},x_n)$$

for every $0 < l < 1$, then the sequence $\{x_n\}$ is a Cauchy sequence.

Proof. For every $n \in \mathbb{N}$ and $x_n, x_{n+1} \in X$, we have

$$\begin{aligned} S(x_n,x_n,x_{n+1}) &\leq lS(x_{n-1},x_{n-1},x_n) \\ &\leq l^2 S(x_{n-2},x_{n-2},x_{n-1}) \\ &\ \vdots \\ &\leq l^n S(x_0,x_0,x_1). \end{aligned}$$

Hence for every $m > n$ and $0 < l < 1$ we have, by the triangle inequality,

$$\begin{aligned} S(x_n,x_n,x_m) &\leq 2\sum_{i=n}^{m-2} S(x_i,x_i,x_{i+1}) + S(x_{m-1},x_{m-1},x_m) \\ &\leq 2[l^n + l^{n+1} + \cdots + l^{m-1}]S(x_0,x_0,x_1) \\ &\leq \frac{2l^n}{1-l} S(x_0,x_0,x_1) \longrightarrow 0. \end{aligned}$$

Therefore, for each $\epsilon > 0$, there exits $n_0 \in \mathbb{N}$, such that, for each $n, m \geq n_0$

$$S(x_n,x_n,x_m) < \epsilon.$$

These show that $\{x_n\}$ is Cauchy sequence in X. □

Proposition 2.1. *Let (X,S) be an S-metric space. Then $\left(X,d,\frac{3}{2}\right)$ is a b-metric space, where*

$$d(x,y)=S(x,x,y),$$

for all $x,y\in X$.

Proof. Since $S(x,x,y)=S(y,y,x)$ it follows that $d(x,y)=d(y,x)$. Further, $d(x,y)\geq 0$ as well as $d(x,y)=0$ if and only if $x=y$. We will prove that

$$d(x,y)\leq \frac{3}{2}\left(d(x,y)+d(y,z)\right).$$

Indeed, according to (S2) we have (for each $x,y,z\in X$)

$$\begin{aligned} d(x,y) &= S(x,x,y)\leq 2S(x,x,z)+S(y,y,z)\\ &= 2d(x,z)+d(y,z), \end{aligned}$$

and

$$\begin{aligned} d(x,y) &= S(y,y,x)\leq 2S(y,y,z)+S(x,x,z)\\ &= 2d(y,z)+d(x,z). \end{aligned}$$

It follows that $d(x,y)\leq \frac{3}{2}[d(x,z)+d(y,z)]$, that is d is one b-metric on the nonempty set X. □

Corollary 2.1. *Let (X,S) be an S-metric space. Then the b-metric d defined in Proposition 2.1 is a continuous function in both variables.*

Proposition 2.2. *Let (X,S) be an S-metric space. If the sequence $\{x_n\}$ in X is S-convergent to x, then $\{x_n\}$ is an S-Cauchy sequence.*

Proof. According to (S2) we have

$$S(x_n,x_n,x_m)\leq 2S(x_n,x_n,x)+S(x_m,x_m,x)\to 0, \text{ when both } n,m\to\infty.$$

□

Proposition 2.3. *Let (X,S) be an S-metric space. If the sequence $\{x_n\}$ in X be S-convergent to x, then x is unique.*

Proof. Let the sequence $\{x_n\}$ S-converges to some $y\in X, y\neq x$. Then, according to the (S2), we have

$$\begin{aligned} S(x,x,y) &\leq 2S(x,x,x_n)+S(y,y,x_n)\\ &= 2S(x_n,x_n,x)+S(x_n,x_n,y)\\ &\to 0+0=0, \quad \text{when} \quad n\to\infty. \end{aligned}$$

Hence, from (S1) follows that $x=y$, a contradiction. □

Definition 2.3. *Let (X,S) be an S-metric space. For $r>0$ and $x\in X$ we define open ball $B_S(x,r)$ and the closed ball $B_S[x,r]$, with center x and radius r, as follows respectively:*

$$\begin{aligned} B_S(x,r) &= \{y\in X: S(y,y,x)<r\},\\ B_S[x,r] &= \{y\in X: S(y,y,x)\leq r\}. \end{aligned}$$

Definition 2.4. *Let (X,S) be an S-metric space. Let τ be the set of all $A\subset X$ with $x\in A$ if and only if there exists an $r>0$ such that $B_S(x,r)\subset A$. Then τ is a topology on X (induced by the S-metric S).*

Proposition 2.4. *Let (X,S) be an S-metric space. If $r>0$ and $x\in X$, then the ball $B_S(x,r)$ is a $\tau-$open subset of X.*

Proof. Let $y\in B_S(x,r)$. Hence $S(y,y,x)<r$. If we set $\delta=S(x,x,y)$ and $r'=\frac{r-\delta}{2}$ then we prove that $B_S(y,r')\subseteq B_S(x,r)$. Let $z\in B_S(y,r')$. Then $S(z,z,y)<r'$. Using (S2) we have

$$S(z,z,x)\leq 2S(z,z,y)+S(x,x,y)<2r'+\delta=r.$$

Hence $B_S(y,r')\subseteq B_S(x,r)$. This means that the ball $B_S(x,r)$ is an $\tau-$open subset of X. □

Lemma 2.4. *Let (X,S) be an $S-$metric space and let $\{x_n\}$ be a sequence in it such that*

$$\lim_{n\to\infty} S(x_{n+1},x_{n+1},x_n)=0.$$

If $\{x_n\}$ is not an S-Cauchy sequence, then there exist an $\varepsilon>0$ and two sequences $\{n_k\}$ and $\{m_k\}$ of positive integers such that the following sequences tend to ε when $k\to\infty$:

$$\{S(x_{m_k},x_{m_k},x_{n_k})\},\ \{S(x_{m_k},x_{m_k},x_{n_k+1})\},\ \{S(x_{m_k+1},x_{m_k+1},x_{n_k+1})\},$$
$$\{S(x_{m_k-1},x_{m_k-1},x_{n_k})\},\ \{S(x_{m_k-1},x_{m_k-1},x_{n_k+1})\},....$$

Proof. If $\{x_n\}$ is not an S-Cauchy sequence, then there exist an $\varepsilon>0$ and two sequences $\{n_k\}$ and $\{m_k\}$ of positive integers such that

$$n_k>m_k>k,\ S(x_{m_k},x_{m_k},x_{n_k-1})<\varepsilon,\ S(x_{m_k},x_{m_k},x_{n_k})\geq\varepsilon \text{ for all } k\in\mathbb{N}.$$

From Lemma 2.1 and (S2),

$$\begin{aligned}\varepsilon &\leq S(x_{m_k},x_{m_k},x_{n_k})=S(x_{n_k},x_{n_k},x_{m_k})\leq 2S(x_{n_k},x_{n_k},x_{n_k-1})+S(x_{m_k},x_{m_k},x_{n_k-1})\\ &< 2S(x_{n_k},x_{n_k},x_{n_k-1})+\varepsilon\to 0+\varepsilon=\varepsilon\end{aligned}$$

when $k\to\infty$. Hence, $S(x_{m_k},x_{m_k},x_{n_k})\to\varepsilon$ as $k\to\infty$.

Similarly,

$$\begin{aligned}S(x_{m_k},x_{m_k},x_{n_k})=S(x_{n_k},x_{n_k},x_{m_k})&\leq 2S(x_{n_k},x_{n_k},x_{n_k+1})+S(x_{m_k},x_{m_k},x_{n_k+1})\\ &=S(x_{m_k},x_{m_k},x_{n_k+1})+2S(x_{n_k+1},x_{n_k+1},x_{n_k}),\end{aligned}$$

as well as

$$\begin{aligned}S(x_{m_k},x_{m_k},x_{n_k+1})&=S(x_{n_k+1},x_{n_k+1},x_{m_k})\\ &\leq 2S(x_{n_k+1},x_{n_k+1},x_{n_k})+S(x_{m_k},x_{m_k},x_{n_k}).\end{aligned}$$

Hence $S(x_{m_k},x_{m_k},x_{n_k+1})\to\varepsilon$ as $k\to\infty$.

Further, we have

$$S(x_{m_k},x_{m_k},x_{n_k+1})\leq 2S(x_{m_k},x_{m_k},x_{m_k+1})+S(x_{n_k+1},x_{n_k+1},x_{m_k+1})$$

and

$$\begin{aligned}S(x_{m_k+1},x_{m_k+1},x_{n_k+1})&\leq 2S(x_{m_k+1},x_{m_k+1},x_{m_k})+S(x_{n_k+1},x_{n_k+1},x_{m_k})\\ &=2S(x_{m_k+1},x_{m_k+1},x_{m_k})+S(x_{m_k},x_{m_k},x_{n_k+1})\end{aligned}$$

from which it follows that $S(x_{m_k+1},x_{m_k+1},x_{n_k+1})\to\varepsilon$ as $k\to\infty$.

Similarly, we further obtain

$$S(x_{m_k},x_{m_k},x_{n_k+1})\leq 2S(x_{m_k},x_{m_k},x_{m_k-1})+S(x_{m_k-1},x_{m_k-1},x_{n_k})$$

$$S(x_{m_k-1},x_{m_k-1},x_{n_k})\leq 2S(x_{m_k-1},x_{m_k-1},x_{m_k})+S(x_{m_k},x_{m_k},x_{n_k}),$$

that is., $S(x_{m_k-1},x_{m_k-1},x_{n_k})\to\varepsilon$ as $k\to\infty$.

Finally, we have

$$\begin{aligned} S\left(x_{m_k-1}, x_{m_k-1}, x_{n_k+1}\right) &\leq 2S\left(x_{m_k-1}, x_{m_k-1}, x_{m_k}\right) + S\left(x_{n_k+1}, x_{n_k+1}, x_{m_k}\right) \\ &= 2S\left(x_{m_k-1}, x_{m_k-1}, x_{m_k}\right) + S\left(x_{m_k}, x_{m_k}, x_{n_k+1}\right) \end{aligned}$$

and

$$\begin{aligned} S\left(x_{m_k}, x_{m_k}, x_{n_k+1}\right) &\leq 2S\left(x_{m_k}, x_{m_k}, x_{m_k-1}\right) + S\left(x_{n_k+1}, x_{n_k+1}, x_{m_k-1}\right) \\ &= 2S\left(x_{m_k}, x_{m_k}, x_{m_k-1}\right) + S\left(x_{m_k-1}, x_{m_k-1}, x_{n_k+1}\right) \end{aligned}$$

that is., $S\left(x_{m_k-1}, x_{m_k-1}, x_{n_k+1}\right) \to \varepsilon$ as $k \to \infty$. □

Analogous to metric spaces, the notion of $S-$compatible pair of self mappings (f, g) has been introduced in the framework of an $S-$metric spaces in the following way.

Definition 2.5. [47] *Let (X, S) be an $S-$metric space. A pair $\{f, g\}$ is said to be S-compatible if and only if* $\lim_{n\to\infty} S\left(fgx_n, gfx_n\right) = 0$, *whenever $\{x_n\}$ is a sequence in X such that* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ *for some $t \in X$.*

Example 2.2. [47] *Let $X = [0, 1]$ be endowed with the $S-$metric $S\left(x, y, z\right) = |x - z| + |y - z|$. Define f, g, R and T on X by*

$$f\left(x\right) = \left(\frac{x}{2}\right)^8, g\left(x\right) = \left(\frac{x}{2}\right)^4, R\left(x\right) = \left(\frac{x}{2}\right)^2 \text{ and } T\left(x\right) = \frac{x}{2}.$$

Then the pairs $\{f, R\}$ and $\{g, T\}$ are S-compatible, but they are not commuting.

Remark 2.1. *It is clear that a pair $\{f, g\}$ is S-compatible if and only if it is $d-$ compatible, where d is the $b-$ metric introduced in Proposition 2.1.*

The following proposition is a nice generalization of the corresponding result from metric spaces in the framework of $S-$metric spaces. Our proof is different from [45].

Proposition 2.5. [45] *Let (X, S) be an S-metric space. If there exists two sequences $\{x_n\}$ and $\{y_n\}$ such that* $\lim_{n\to\infty} S\left(x_n, x_n, y_n\right) = 0$ *whenever $\{x_n\}$ is a sequence in X such that* $\lim_{n\to\infty} x_n = t$ *for some $t \in X$, then* $\lim_{n\to\infty} y_n = t$.

Proof. For the proof we use the b-metric introduced in 2.1. Therefore we have

$$\begin{aligned} \frac{2}{3}d\left(y_n, t\right) &\leq d\left(y_n, x_n\right) + d\left(x_n, t\right) \\ &= S\left(x_n, x_n, y_n\right) + S\left(x_n, x_n, t\right) \\ &\to 0 + 0 = 0. \end{aligned}$$

The result now follows since $d\left(y_n, t\right) = S\left(y_n, y_n, t\right)$. □

If (X, S) is an $S-$metric space then $X \times X$ can also be endowed also with some new $S-$metric. Namely, we have the following result.

Proposition 2.6. *Let (X, S) be an $S-$ metric space. Define $S_+ : X^2 \times X^2 \times X^2 \to [0, \infty)$ and $S_{\max} : X^2 \times X^2 \times X^2 \to [0, \infty)$ as*

$$\begin{aligned} S_+\left(\left(x, y\right), \left(u, v\right), \left(p, q\right)\right) &= S\left(x, u, p\right) + S\left(y, v, q\right) \text{ and} \\ S_{\max}\left(\left(x, y\right), \left(u, v\right), \left(p, q\right)\right) &= \max\left\{S\left(x, u, p\right), S\left(y, v, q\right)\right\}. \end{aligned}$$

Then, $\left(X^2, S_+\right)$ and $\left(X^2, S_{\max}\right)$ are two new $S-$metric spaces.

Proof. Let $S_+((x,y),(u,v),(p,q)) = 0 \Leftrightarrow S(x,u,p) = 0$ and $S(y,v,q) = 0 \Leftrightarrow x = u = p$ and $y = v = q \Leftrightarrow (x,y) = (u,v) = (p,q)$. Hence (S$_+$1) holds. We will further prove that

$$\begin{aligned} & S_+((x,y),(u,v),(p,q)) \\ \leq\ & S_+((x,y),(x,y),(a,b)) + S_+((u,v),(u,v),(a,b)) + S_+((p,q),(p,q),(a,b)), \end{aligned}$$

for all $(x,y),(u,v),(p,q),(a,b) \in X^2$.

Indeed, we have that

$$\begin{aligned} & S_+((x,y),(u,v),(p,q)) = S(x,u,p) + S(y,v,q) \\ \leq\ & [S(x,x,a) + S(u,u,a) + S(p,p,a)] + [S(y,y,b) + S(v,v,b) + S(q,q,b)] \\ =\ & [S(x,x,a) + S(y,y,b)] + [S(u,u,a) + S(v,v,b)] + [S(p,p,a) + S(q,q,b)] \\ & = S_+((x,y),(x,y),(a,b)) + S_+((u,v),(u,v),(a,b)) + S_+((p,q),(p,q),(a,b)). \end{aligned}$$

That is, (S$_+$2) holds. This means that (X^2, S_+) is a new $S-$metric space if (X,S) is one.

For the function $S_{\max}$ we have the following. Define $S_{\max}((x,y),(u,v),(p,q)) = 0$. This is equivalent to $S(x,u,p) = 0$ and $S(y,v,q) = 0$; that is, $(x,y) = (u,v) = (p,q)$. Hence, $(S_{\max}1)$ holds. Further, we obtain

$$\begin{aligned} & S_{\max}((x,y),(u,v),(p,q)) = \max\{S(x,u,p), S(y,v,q)\} \\ \leq\ & \max\{S(x,x,a) + S(u,u,a) + S(p,p,a), S(y,y,b) + S(v,v,b) + S(q,q,b)\} \\ \leq\ & \max\{S(x,x,a), S(y,y,b)\} + \max\{S(u,u,a), S(v,v,b)\} \\ +\ & \max\{S(p,p,a), S(q,q,b)\} \\ =\ & S_{\max}((x,y),(x,y),(a,b)) + S_{\max}((u,v),(u,v),(a,b)) + S_{\max}((p,q),(p,q),(a,b)), \end{aligned}$$

that is, $(S_{\max}2)$ holds.

The proof of Proposition 2.6 is complete finished. □

For more information about S-metric space, we refer the reader to ([5]-[9], [15], [16], [22], [26], [27], [31], [32], [34], [38], [46]-[48]).

In this section we use the concept of simulation functions (see [2], [17], [20], [21], [28]- [30]) to present a very general kind of contraction on $S-$metric spaces, and we prove related existence and uniqueness coincidence point results.

Definition 2.6. *A simulation function is a mapping* $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *satisfying the following conditions:*

(ζ_1) $\zeta(0,0) = 0$;

(ζ_2) $\zeta(t,s) < s - t$ *for all* $t,s > 0$;

(ζ_2) *If* $\{t_n\},\{s_n\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, *then* $\limsup_{n\to\infty} \zeta(t_n,s_n) < 0$.

For more details see [20].

Let $\mathcal{Z}$ be the family of all simulation functions $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$.

Next we give some examples of the simulation function.

Example 2.3. *Let* $\zeta_i : [0,\infty) \times [0,\infty) \to \mathbb{R}, i = 1,2,3,4$ *be defined by*

(i) $\zeta_1(t,s) = \psi(s) - \phi(t)$ *for all* $t,s \in [0,\infty)$, *where* $\phi,\psi : [0,\infty) \to [0,\infty)$ *are two continuous functions such that* $\psi(t) = \phi(t) = 0$ *if and only if* $t = 0$ *and* $\psi(t) < t \leq \phi(t)$ *for all* $t > 0$.

(ii) $\zeta_2(t,s) = s - \frac{f(t,s)}{g(t,s)} \cdot t$ *for all* $t,s \in [0,\infty)$, *where* $f,g : [0,\infty) \to [0,\infty)$ *are two continuous functions with respect to each variable such that* $f(t,s) > g(t,s)$ *for all* $t,s > 0$.

(iii) $\zeta_3(t,s) = s - \varphi(s) - t$ *for all* $t,s \in [0,\infty)$, *where* $\varphi : [0,\infty) \to [0,\infty)$ *is a continuous function such that* $\varphi(t) = 0$ *if and only if* $t = 0$.

(iv) $\zeta_4(t,s) = \frac{s}{s+1} - t$ *for all* $t,s \in [0,\infty)$.

Then ζ_i for $i = 1,2,3,4$ are simulation functions.

Definition 2.7. *Let* (X,S) *be an* $S-$*metric space and let* $f,g : X \to X$ *be self-mappings. We say that* f *is a* $(\mathcal{Z},g)-$*contraction if there exists a* $\zeta \in \mathcal{Z}$ *such that*

$$\zeta(S(fx,fx,fy), S(gx,gx,gy)) \geq 0 \text{ for all } x,y \in X \text{ such that } gx \neq gy. \tag{1}$$

Remark 2.2. *If* f *is* $(\mathcal{Z},g)-$ *contraction with respect to some* $\zeta \in \mathcal{Z}$, *then*

$$S(fx,fx,fy) < S(gx,gx,gy) \text{ for all } x,y \in X \text{ such that } gx \neq gy, \tag{2}$$

To prove (2) assume that $gx \neq gy$. Then $S(gx,gx,gy) > 0$. If $fx = fy$, then $S(fx,fx,fy) = 0 < S(gx,gx,gy)$. If $fx \neq fy$, then $S(fx,fx,fy) > 0$, and applying (ζ_2) and (1), we have that

$$0 \leq \zeta(S(fx,fx,fy), S(gx,gx,gy)) < S(gx,gx,gy) - S(fx,fx,fy),$$

and (2) holds.

Now we prove that coincidence points of $(\mathcal{Z},g)-$contractions have the same image by g and f.

Proposition 2.7. *If* f *is a* $(\mathcal{Z},g)-$*contraction in an S-metric space* (X,S) *and* $x,y \in X$ *are coincidence points of* f *and* g, *then* $fx = gx = gy = fy$.

Proof. Suppose that $gx \neq gy$. Then $S(gx,gx,gy) > 0$. By (1), it follows that

$$0 \leq \zeta(S(fx,fx,fy), S(gx,gx,gy)) = \zeta(S(gx,gx,gy), S(gx,gx,gy)) < 0$$

(by (ζ_2)), a contradiction. □

Definition 2.8. *Given two self-mappings* $f,g : X \to X$ *and a sequence* $\{x_n\}_{n\geq 0} \subseteq X$, *we say that* $\{x_n\}$ *is a Picard-Jungck sequence of the pair* (f,g) *(based on* x_0*) if* $fx_n = gx_{n+1}$ *for all* $n \geq 0$. *We say that* X *satisfies the* $CLR_{(f,g)}-$*property at a point* $x_0 \in X$, *if there exists on* X *a Picard-Jungck sequence of* (f,g)*based on* x_0.

Two examples:

1. It is well known that, if f and g are two self-mappings such that $f(X) \subseteq g(X)$, then there exists a Picard-Jungck sequence of (f,g) based on any point $x_0 \in X$. In order words, if $f(X) \subseteq g(X)$, then X satisfies the $CLR_{(f,g)}-$property at each point $x \in X$. The converse it is not true in general.

2. If $g = I_X$ is the identity mapping on X, then there exists a unique Picard-Jungck sequence of (f,g) based at each $x_0 \in X$, which is given by $x_{n+1} = fx_n$ for all $n \geq 0$. Therefore, X satisfies the $CLR_{(f,g)}-$property at every point.

Definition 2.9. *Let* f,g *be mappings on a S-metric space* (X,S). *We say that* f *and* g *are S-compatible if* $\lim_{n\to\infty} S(fgx_n, gfx_n) = 0$ *for all sequence* $\{x_n\} \subseteq X$ *such that the sequence* $\{gx_n\}$ *and* $\{fx_n\}$ *are S-convergent and have the same* $S-$*limit.*

Now, we can prove the first new results in this framework.

Theorem 2.1. *Let* f *be a* $(\mathcal{Z},g)-$*contraction in a* S*-metric space* (X,S) *and suppose that there exists a Picard-Jungck sequence* $\{x_n\}$ *of* (f,g). *Also, suppose that at least one of the following conditions holds.*

(a) $g(X)$ *or* $f(X)$ *is* S*-complete.*

(b) (X, S) is S-complete and f and g are S -continuous and $S-$compatible.

(c) (X, S) is a S-complete and f and g are $S-$ continuous and commuting.

Then f and g have a coincidence point. Furthermore, either the sequence $\{gx_n\}$ contains a coincidence point of f and g or, at least one of the following properties holds.

1. In case (a), the sequence $\{gx_n\}$ converges to $u \in g(X)$ and any point $v \in X$ such that $u = gv$ is a coincidence point of f and g.

2. In cases (b) and (c) the sequence $\{gx_n\}$ converges to a coincidence point of f and g.

3. Conclusion

The paper describes S-metric spaces and simulation functions in the context of S-metric spaces.

4. Acknowledgement

The first author is thankful to the Ministry of Education, Sciences and Technological Development of Serbia (project no. 174009) and by Provincial Secretariat for higher education and scientific research, Province of Vojvodina, Serbia, (project no. 142-451-2838/2017-01).

References

[1] Aghajani, A., Abbas, M., Roshan, J. R., (2014), Common fixed point of generalized weak contractive mappings in partially ordered b. metric spaces, Math. Slovaca, 4, pp.941-960.

[2] Argoubi, H., Samet, B., Vetro, C., (2015), Nonlinear contractions involving simulation functions in a metric spacer with a partial order, J. Nonlinear Sci. Appl., 8, pp.1082-1094.

[3] Bakhtin, I. A., (1989), The contraction mapping principle in almost metric spaces, Funct. Anal., 30, Unianowsk, Gos. Ped. Inst., pp.26-37.

[4] Boriceanu, M., (2009), Strict fixed point theorems for multivalued operators in b-metric spaces, Int. J. Mod. Math., 4, pp.285-301.

[5] Chaipornjareansri, S., (2016), Fixed point theorems for F_w-contractions in complete S-metric spaces, Thai Journal of Mathematics: Special Issue on: Advances in fixed point theory towards real world optimization problems, pp.98-109.

[6] Czerwik, S., (1993), Contraction mappings in b-metric spaces, Acta Mathematica et Informatica Universitatis Ostraviensis, 1, pp.5-11.

[7] Dhage, B. C., (1992), Generalized metric spaces and mapping with fixed points, Bull. Cal. Math. Soc., 84, pp.329-336.

[8] Dung, N.V., Hieu, N. T., Radojević, S., (2014), Fixed point theorems for g-monotone maps on partially ordered S-metric spaces, Filomat 28(9), pp.1885-1898.

[9] Dung, N. V., (2013), On coupled common fixed points for mixed weakly monotone maps in partially ordered $S-$ metric spaces, Fixed Point Theory Appl., 2013:48.

[10] Dung, N. V., (2014), Remarks on quasi-metric spaces, Miskolc Mathematical Notes, 15(2), pp. 401-422.

[11] Dung, N. V., Hieu, N. T., Ly, N. T. T., Thinh, V. D., (2013), Remarks on the fixed point problem of $2-$metric spaces, Fixed Point Theory Appl., 2013:167.

[12] Gahler, S., (1963), $2-$metrische Raume und ihre topologische Struktur, Mathematische Nachrichten, 26, pp.115-148.

[13] Gahler, S., (1966), Zur geometric 2-metrische raume, Revue Roumaine de Mathématiques Pures et Appliquées", 40, pp.664-669.

[14] Gajić, Lj., Arshad, M., Khan, S.U., Rahman, L., (2017), Some new fixed point results in ultra metric space, TWMS J. Pure Appl. Math., 8(1), pp.33-42.

[15] Gupta, A., (2013), Cyclic contraction on $S-$ metric space, Int. J. Anal. Appl., 3(2), pp.119-130.

[16] Gupta, V., Deep, R., (2015), Some coupled fixed point theorems in partially ordered $S-$ metric spaces, Miskloc Mathematical Notes, 16(1), pp.181-194.

[17] Hierro, A-F.R.L., Karapnar, E., Hierro, C.R.-L., Moreno, J.M., (2015), Coincidence point theorems on metric spaces via simulation functions, J. Comput. Appl. Math., 275, pp.345-355.

[18] Hussain, N., Parvaneh, V., Roshan, JR., Kadelburg, Z., (2013), Fixed points of cyclic weakly (ψ, ϕ, L, A, B)-contractive mappings in ordered b.metric spaces with applications, Fixed Point Theory Appl., 2013, Article ID 256.

[19] Jleli, M., Samet, B., (2012), Remarks on G-metric spaces and fixed point theorems, Fixed Point Theory Appl., 2012:201.

[20] Khojasteh, F., Shukla, S., Radenović, S., (2015), A new approach to study of fixed point theory for simulation functions, Filomat 29(6), pp. 1189-1194.

[21] Liu, X.-L., Ansari, A.H., Chandok, S., Radenović, S., (2018), On some results in metric spaces using auxiliary simulation functions via new functions, J. Comput. Anal. Appl., 24(6), pp.1103-1114.

[22] Mlaiki, N. M., (2015), $\alpha - \psi-$ contractive mapping on S-metri cspace, Math. Sci. Lett., 4(1), pp.9-12.

[23] Mohanta, S. K., (2015), Some fixed point theorems using wt-distance in b-metric spaces, Fasciculi Math., 54, pp. 125-140.

[24] Mustafa, Z., Sims, B., (2003), Some remarks concerning D-metric spaces, Proceedings of the International Conferences on Fixed Point Theory and Applications, Valencia (Spain), pp.189-198.

[25] Mustafa, Z., Sims, B., (2006), "A new approach to generalized metric spaces," Journal of Nonlinear and Convex Analysis, 7(2), pp.289-297.

[26] Ozgur, N. Y., Tas, N., (2017), Some fixed point theorems on $S-$metric spaces, Matematički Vesnik, 69(1), pp.39-52.

[27] Prudhvi, K., (2016), Some fixed point results in $S-$metric spaces, J. Math. Sci. Appl., 4(1), pp.1-3.

[28] Rad, S.G., Radenović, S., Dolićanin-Đekić, D., A shorter and simple approach to study fixed point results via b-simulation functions, to appear in Iranian Journal of Mathematical Sciences and Informatics.

[29] Radenović, S., Vetro, F., Vujaković, J., (2017), An alternative and easy approach to fixed point results via simulation functions, Demonstratio Mathematica, 50, pp.224-231.

[30] Radenović, S., Chandok, S., Simulation type functions and coincidence point results, to appear in Filomat.

[31] Raj, H., Hooda, N., (2014), Coupled fixed point theorems in S-metrioc spaces with mixed G-monotone property, Inter. J. Emerging Trends in Engin. Develop., 4(4), pp.68-81.

[32] Rohen, Y., Došenović, T., Radenović, S., (2017), A note on the paper "A fixed point theorems in S_b- metric spaces" Filomat, 31(11), pp.3335-3346.

[33] Roshan, J.R., Parvaneh, V., Sedghi, S., Shobkolaei, N., Shatanawi, W., (2013), Common fixed point of almost generalized $(\omega, \varphi)_s -$ contractive mappings in ordered b-metric spaces, Fixed Point Theory Appl., 2013:159.

[34] Sedghi, S., Rezaee, M. M., Došenović, T., Radenović, S., (2016), Common fixed point theorems for contractive mappings satisfying $\Phi-$maps in $S-$metric spaces, Acta Univ. Sapientiae, Mathematica, 8(2), pp.298-311.

[35] Sedghi, S., Dung, N.V., (2014), Fixed point theorems on S-metric spaces, Matematički Vesnik, 66(1), pp. 113-124.

[36] Sedghi, S., Shobe, N., Aliouche, A., (2012), A generalization of fixed point theorems in $S-$metric spaces, Matematički Vesnik, 64(3), pp.258-266.

[37] Sedghi, S., Shobe, N., Došenović, T., (2015), Fixed point results in S-metric spaces, Nonlinear Functional Analysis Appl., 20(1), pp.55-67.

[38] Sedghi, S., Gholidahneh, A., Došenović, T., Esfahani, J., Radenović, S., (2016), Common fixed point of four maps in S_b- metric spaces, J. Linear Topol. Algebra, 05(02), pp.93-104.

[39] Sedghi, S., Rao, K.P.R., Shobe, N., (2007), Common fixed point theorems for six weakly compatible mappings in D^*-metric spaces, International Journal of Mathematical Sciences, 6(2), pp.225-237.

[40] Sedghi, S., Shobe, N., Zhou, H., (2007), A common fixed point theorem in D^*-metric spaces, Fixed point Theory and Applications, Volume 2007, Article ID 27906, 13 pages.

[41] Sedghi, S., Turkoglu, D., Shobe, N., Sedghi, S., (2009), Common Fixed Point Theorems for Six Weakly Compatible Mappings in D^*-Metric Spaces, Thai Journal of Mathematics, 7(2), pp.381-391.

[42] Sedghi, S., Shobe, N. Sedghi, S., (2008), Common fixed point theorems for two mappings in D^*-metric spaces, Journal of Prime Research in Mathematics, 4, pp.132-142.

[43] Sedghi, S., Shobe, N., (2011), A common unique random fixed point theorems in $S-$ metric spaces, Journal of Prime Research in Mathematics, 7, pp.25-34.

[44] Sedghi, S., Altun, I., Shobe, N., Salahshour, M. A., (2014), Some Properties of $S-$metric Spaces and Fixed Point Results, Kyungpook Math. J., 54, pp.113-122.
http://dx.doi.org/10.5666/KMJ. 2014. 54. 1. 113.

[45] Sedghi, S., Shobkolaei, N., Shahraki, M., Došenović, T., Common fixed point of four maps in $S-$ metric spaces, submitted.

[46] Sharma, A., Tiwari, M., Bhardwaj, R., (2016), Pseudo-$S-$ metric spaces and pseudo-$S-$metric product spaces, International Journal of Theoretical and Applied Sciences, Special Issue-NCRTAST, 8(1), pp.139-141.

[47] Singh, A., Hooda, N., (2014), Coupled fixed point theorems in $S-$metric spaces, International Journal of Mathematics and Statistics Invention, 2(4), pp.33-39.

[48] Souayah, N., Mlaiki, N., (2016), A fixed point theorem in S_b- metric spaces, J. Math. Computer Sci., 16, pp.131-139.

[49] Wilson, W.A., (1931), On semi-metric spaces, American Journal of Mathematics, 53(2), pp.361-373.

Tatjana Došenović was born in 1973 in Novi Sad, Serbia. She graduated in 1997, and defended Ph.D. thesis in 2002 from Faculty of Science in Novi Sad. From 1998 she started to work at Faculty of Technology, University of Novi Sad, and from 2009 she is an associate professor at the same faculty. Her research interests are functional analysis, especially, the theory of fixed points in metric space, fuzzy metric space and abstract metric spaces.

Stojan Radenović, for a photograph and biography, see TWMS J. Pure Appl. Math., V.5, N.1, 2014, p.13.

Shaban Sedghi received his B.Sc and M.Sc from University of Ferdosse Mashhad (Iran) and his Ph.D. degree in Pure Mathematics from Islamic Azad University, Tehran (Iran) in 1998. From 1988 he started to work at Department of Mathematics, Qaemshahr University, and from 2007 he is an associate professor at the same University. His research interest topics are fixed point theory, group theory and fuzzy theory.